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# SUPERCRITICAL AND SUBCRITICAL HOPF BIFURCATIONS IN A STOCHASTICALLY EXCITED SYSTEM

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## 1. INTRODUCTION

Bifurcation is an important non-linear phenomenon that can occur in many engineering dynamic systems. A system is said to be very "weak" if the bifurcation parameter is in the neighborhood of the bifurcation point. In such a system, any small perturbation can change its dynamical properties. There are books [1–3], for example, that deal with bifurcations in systems disturbed by deterministic excitations. The study of bifurcations of systems under stochastic excitations has only been pursued in the past decade or so. Typical examples are reported in references [4–11]. In reference [4], it was incorrectly concluded that the bifurcation point depends on the order of statistical moment of response. Mathematically, the techniques [8–10] based on the largest Lyapunov exponents of linearized oscillators seem to be the most rigorous ones. However, the largest Lyapunov exponent of a linearized oscillator provides only the necessary condition for bifurcation analysis of non-linear oscillators and, therefore, additional tools are required. The technique based on using the largest Lyapunov exponent of the linearized system and the stochastic centre manifold theorem requires the determination of the stochastic centre manifold [8]. The latter, in turn, calls for the computation of the cocycle of non-linear diffeomorphisms, which remains the major difficulty for general two- and higher-dimensional problems. On the other hand, the technique based on the largest Lyapunov exponent and the method of stochastic averaging of Khas'minskii, applied to two first order equations describing the response and energy of the oscillators, assumes the existence of a stationary probability density distribution [9]. In reference [11], application of the theory of moment Lyapunov exponents [12] has been made for the bifurcation analysis. It should be noted that the theory of Lyapunov exponents hinges around Oseledec's ergodic theorem [13]. The validity of the latter theorem in the theory of moment Lyapunov exponents for non-linear system is not clearly shown. As the moments of responses of systems under stationary excitations are stationary and are independent of time, it seems that the existence of the limit in time in the definition of a moment Lyapunov exponent for non-linear systems cannot be established. Furthermore, while the Lyapunov exponents are analogues to the eigenvalues of deterministic systems and therefore have ample physical meaning, moment Lyapunov exponents are difficult to interpret physically. For linearized systems the moment Lyapunov exponents have been regarded, roughly speaking, as the exponential growth or decay rate. On the other hand, for non-linear systems such an interpretation is far from satisfactory.

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Consequently, in the investigation reported here the Hopf bifurcations of a general non-linear system that is disturbed by stationary stochastic excitations are studied. We focus our study on the cases in which the system undergoes supercritical and subcritical Hopf bifurcations when the stochastic excitations are removed. Our analysis is based mainly on the perturbation method [2], the stochastic averaging of Stratonovich [14], some results from the theory of singularity and group theory [3], and the results of the stability analysis of Brouwers [15]. This novel approach is different from those based on the application of the largest Lyapunov exponent of the system.

## 2. STATEMENT OF THE PROBLEM

Consider the equation

$$\mathrm{d}y/\mathrm{d}t = F(y, \lambda) + \sqrt{\varepsilon}[A_r f_r(t)]y, \qquad r = 1, 2, \ldots, n, \tag{1}$$

where $y = (y_1, y_2)$ is a two-dimensional state vector, $F$: $R^2 \times R \to R^2$ is $C^\infty$ and is defined on a neighborhood of the origin (0, 0); $A_r(\lambda)$ ($r = 1, 2, \ldots, n$) are constant $2 \times 2$ matrices which are $C^\infty$ in $\lambda, f_r(t)$ are uncorrelated broadband stationary stochastic processes with zero mean values, $|\varepsilon| \ll 1$ and $f_r(t)$ has an arbitrary smoothly varying spectral density function. For the unperturbed system, that is, $\varepsilon = 0$, we assume that (1) $F(0, \lambda) = 0$ and $A(\lambda) = D_y F(0, \lambda)$, and (2) $A(\lambda)$ has eigenvalues $\sigma(\lambda) + \mathrm{i}\omega(\lambda)$ and $\sigma(0) = 0$, $\omega(0) = 1$ and $\sigma'(0) \neq 0$, in which the prime denotes differentiation with respect to $\lambda$.

*Theorem* 1 [3]. Under assumptions (1) and (2), if $r_z \neq 0$ then the bifurcation solutions of the unperturbed system are $Z_2$-equivalent to $\delta_1 x^3 - \delta x \lambda$, where $\delta_1 = \operatorname{sgn} r_z$, $\delta = \operatorname{sgn} \sigma'(0)$,

$$r_z = -\tfrac{1}{4} \operatorname{Re} \{\bar{d}^{\mathrm{T}} \cdot [d^2F(c, a_0) + d^2F(\bar{c}, a_2) + \tfrac{1}{4} d^3F(c, c, \bar{c})\}, \tag{2}$$

$$Ac = \mathrm{i}c, \qquad A^{\mathrm{T}}d = -\mathrm{i}d, \qquad \bar{d}^{\mathrm{T}}c = 2, \tag{3}$$

$$Aa_0 = -\tfrac{1}{2} d^2F(c, \bar{c}), \quad (A - 2\mathrm{i}I)a_2 = -\tfrac{1}{4} d^2F(c, c), \qquad A \equiv A(0), \tag{4, 5}$$

and where the overbar denotes the complex conjugate, the superscript T denotes the transpose, and the remaining symbols are defined in reference [3].

We study the effect of the stationary stochastic excitations on a system with Hopf bifurcations. It is well known that there are two types of Hopf bifurcation in the system defined by equation (1) when the excitations are removed [3]. These are the supercritical and subcritical Hopf bifurcations [2, 3]. An important question to ask at this stage is as follows: For a system disturbed by stochastic excitations, do we have the aforementioned types of bifurcations or not? We shall attempt to answer this question in the following sections.

## 3. ANALYSIS

From equation (1), when $\varepsilon = 0$, we have

$$\dot{y} = F(y, \lambda), \tag{6}$$

where the overdot denotes differentiation with respect to time $t$. Expanding $F(y, \lambda)$ in a Taylor's series about the point (0, 0), and representing the second and third homogeneous polynomials as $H_2(y)$ and $H_3(y)$, we have

$$\dot{y} = Ay + \lambda \begin{bmatrix} \sigma'(0) & \omega'(0) \\ -\omega'(0) & \sigma'(0) \end{bmatrix} y + H_2(y) + H_3(y) + o(y^3, \lambda), \tag{7}$$

where $A = A(0)$, as defined in equation (5), and

$$H_2(y) = \begin{Bmatrix} B_{ij}^{(1)} y_i y_j \\ B_{ij}^{(2)} y_i y_j \end{Bmatrix}_{i \leqslant j} \qquad (i, j = 1, 2),$$

$$H_3(y) = \begin{Bmatrix} C_{ijk}^{(1)} y_i y_j y_k \\ C_{ijk}^{(2)} y_i y_j y_k \end{Bmatrix}_{i \leqslant j \leqslant k} \qquad (i, j, k = 1, 2),$$

and where $B_{ij}^{(1)}$, $B_{ij}^{(2)}$, $C_{ijk}^{(1)}$ and $C_{ijk}^{(2)}$, are constants.

By making use of equations (2)–(5) and performing some lengthy algebraic manipulations, one obtains

$$\begin{aligned} r_z = & -\tfrac{1}{8}(2B_{11}^{(1)}B_{11}^{(2)} - B_{11}^{(1)}B_{12}^{(1)} - B_{12}^{(1)}B_{22}^{(1)} - 2B_{22}^{(1)}B_{22}^{(2)} \\ & + B_{11}^{(2)}B_{12}^{(2)} + B_{12}^{(2)}B_{22}^{(2)}) \\ & -\tfrac{1}{8}(3C_{111}^{(1)} + C_{112}^{(2)} + C_{122}^{(1)} + 3C_{222}^{(2)}). \end{aligned} \tag{8}$$

Equations (8) is the coefficient of the non-linear term in the system defined by equation (6).

## 4. STANDARD FORM

In order to obtain the standard form we shall rescale the state vector $y$. Setting $x = [x_1 \quad x_2]^{\mathrm{T}}$, $x_1 = \sqrt{\varepsilon} y_1$, $x_2 = \sqrt{\varepsilon} y_2$, and $\lambda = \varepsilon\eta$, from equations (1) and (2) we have

$$\dot{x} = Ax + \varepsilon\eta \begin{bmatrix} \sigma'(0) & \omega'(0) \\ -\omega'(0) & \sigma'(0) \end{bmatrix} x + H_2(x) + H_3(x) + \sqrt{\varepsilon}(A_r f_r(t))x + o(\varepsilon), \tag{9}$$

where

$$A_r = A_r(0) = \begin{bmatrix} A_{r11} & A_{r12} \\ A_{r21} & A_{r22} \end{bmatrix}.$$

We further apply the transformation

$$x_1 = a \sin \Phi, \qquad x_2 = a \cos \Phi, \qquad \Phi = t + \phi \tag{10}$$

and assume that the amplitude $a$ and the phase $\Phi$ are "slowly varying" with respect to $t$. Then, by equations (9) and (10), one can show that

$$\dot{a} = \varepsilon\eta\sigma'(0)a + \sin \Phi G_1 + \cos \Phi G_2 + o(\varepsilon), \tag{11a}$$

$$a\dot{\phi} = \varepsilon\eta\omega'(0)a + \cos \Phi G_1 - \sin \Phi G_2 + o(\varepsilon), \tag{11b}$$

where

$$G_1 = \sqrt{\varepsilon} B_{ij}^{(1)} x_i x_j + \varepsilon C_{ijk}^{(1)} x_i x_j x_k + \sqrt{\varepsilon}(A_{r11}x_1 + A_{r12}x_2) f_r(t),$$

$$G_2 = \sqrt{\varepsilon} B_{ij}^{(2)} x_i x_j + \varepsilon C_{ijk}^{(2)} x_i x_j x_k + \sqrt{\varepsilon}(A_{r21}x_1 + A_{r22}x_2) f_r(t).$$

As the excitations in equation (11) have a common factor $\sqrt{\varepsilon}$, they effectively affect the corresponding deterministic system just to order $\varepsilon$ or higher. Therefore, in considering the stochastic effect we at least approximate the deterministic system up to order $\varepsilon$. In other words, we should consider the second approximation to the deterministic system. On the other hand, as the amplitude and phase are "slowly varying", the approximation should not contain oscillatory terms. Thus, we need to find the second approximation to the terms in equation (11). Before proceeding further, we express equation (11), with the forcing terms disregarded at this stage, as

$$\dot{a} = \varepsilon\eta\sigma'(0)a + \varepsilon a^3 V_a + \sqrt{\varepsilon} a^2 U_a = \sqrt{\varepsilon} G(a, \Phi), \tag{12a}$$

$$\dot{\phi} = \varepsilon\eta\omega'(0) + \varepsilon a^2 V_\phi + \sqrt{\varepsilon} a U_\phi = \sqrt{\varepsilon} H(a, \Phi), \tag{12b}$$

where $U_a$, $V_a$, $U_\phi$ and $v_\phi$ have been given elsewhere and are not included here for brevity.

Now, we return to the approximation. According to the method of Stratonovich [14] we set

$$a = a^* + \sqrt{\varepsilon} p(a^*, \Phi^*), \qquad \Phi = \Phi^* + \sqrt{\varepsilon} q(a^*, \Phi^*), \qquad \Phi^* = t + \phi^*. \quad \text{(13a, b)}$$

The functions $p$ and $q$ are chosen in such a way that the equations for the new amplitude $a^*$ and phase $\Phi^*$, which are equivalent to equation (12), contain no oscillatory terms $\sin k\Phi^*$ and $\cos k\Phi^*$ ($k = 1, 2, \ldots$).

By the transformation defined in equation (13) and some lengthy algebraic manipulations, one can obtain

$$\dot{a}^* = \varepsilon[\sigma'(0)\eta a^* - r_z(a^*)^3] + o(\varepsilon), \qquad \dot{\phi}^* = \varepsilon[\omega'(0)\eta - s(a^*)^2] + o(\varepsilon), \quad \text{(14a, b)}$$

where $r_z$ is given by equation (8) and

$$\begin{aligned} s = -\tfrac{1}{8}[&(B_{11}^{(1)} + B_{22}^{(1)})(B_{12}^{(2)} - B_{11}^{(1)} - 3B_{22}^{(1)}) + (B_{11}^{(2)} + B_{22}^{(2)})(B_{12}^{(1)} - B_{22}^{(2)} - 3B_{11}^{(2)}) \\ &+ \tfrac{1}{3}(B_{11}^{(1)} - B_{22}^{(1)} - B_{12}^{(2)})(B_{22}^{(1)} - B_{11}^{(1)} + B_{12}^{(2)}) + \tfrac{1}{3}(B_{22}^{(2)} - B_{11}^{(2)} - B_{12}^{(1)})(B_{11}^{(2)} - B_{22}^{(2)} + B_{12}^{(1)}) \\ &+ (3C_{222}^{(1)} + C_{112}^{(1)} - C_{122}^{(2)} - 3C_{111}^{(2)})]. \end{aligned}$$

Although the expressions $r_z$ in equation (8) and $s$ in equation (14) tally with $R$ and $S$ in equation (5) of reference [4], it is interesting to note that equation (14) is in terms of second order approxiamtion $a^*$, while equation (5) in reference [4] is in terms of first order approximation $a$.

We can now include the forcing functions of equation (11) into equation (14) and disregard the asterisks in equation (14) for simplicity:

$$\dot{a} = \varepsilon[\sigma'(0)\eta - r_z a^2]a + \tfrac{1}{2}\sqrt{\varepsilon} a G_{rr} f_r(t), \tag{15a}$$

$$\dot{\phi} = \varepsilon[\omega'(0)\eta - s a^2] + \tfrac{1}{2}\sqrt{\varepsilon} H_{rr} f_r(t), \tag{15b}$$

where

$$G_{rr} = A_{r11} + A_{r22} + (A_{r12} + A_{r21})\sin 2\Phi + (A_{r22} - A_{r11})\cos 2\Phi,$$

$$H_{rr} = A_{r12} - A_{r21} + (A_{r11} - A_{r22})\sin 2\Phi + (A_{r12} + A_{r21})\cos 2\Phi.$$

Equation (15) is the required standard form and is equivalent to equation (1) to $o(\varepsilon)$.

## 5. ITO AND FOKKER–PLANCK EQUATIONS

In equation (15) the characteristic parameters for Hopf bifurcations of the deterministic system are included. Thus, we can use the stochastic averaging method for equation (15). Under the hypotheses that the processes $(a, \phi)$ converge weakly to diffusive Markov processes governed by a pair of Ito's equations as $\varepsilon \to 0$, we obtain

$$\mathrm{d}a = \varepsilon m_a\,\mathrm{d}t + \sqrt{\varepsilon}\delta_{1j}\,\mathrm{d}w_j, \qquad \mathrm{d}\phi = \varepsilon n_\phi\,\mathrm{d}t + \sqrt{\varepsilon}\delta_{2j}\,\mathrm{d}w_j, \quad \text{(16a, b)}$$

where $w_j$, $j = 1, 2$, are independent Wiener processes, each having a spectral density of $1/(2\pi)$. The matrix $[\delta] = (\delta_{ij})_{2\times 2}$:

$$m_a = [\sigma'(0)\eta - r_z a^2]a + \tfrac{1}{4}a[(A_{r11} + A_{r22})^2 S_r(0) + \tfrac{3}{2}(A_{r12} + A_{r21})^2 S_r(2) + \tfrac{3}{2}(A_{r22} - A_{r11})^2 S_r(2)],$$

where $S_r(\omega) = \int_0^\infty \langle f_r(t) \quad f_r(t+\tau)\rangle \cos\omega\tau \, \mathrm{d}\tau$, in which the angular brackets denote the mathematical expectation,

$$n_\phi = \omega'(0)\eta - sa^2 + \tfrac{1}{4}[(A_{r11} - A_{r22})^2 + (A_{r12} + A_{r21})^2]\psi_r(2),$$

where

$$\psi_r(\omega) = \int_0^\infty \langle f_r(t) \quad f_r(t+\tau)\rangle \sin\omega\tau \, \mathrm{d}\tau,$$

$$([\delta][\delta]^{\mathrm{T}})_{11} = (a^2/4)[2(A_{r11} - A_{r22})^2 S_r(0) + (A_{r12} + A_{r21})^2 S_r(2) + (A_{r22} - A_{r11})^2 S_r(2)],$$

$$([\delta][\delta]^{\mathrm{T}})_{12} = (a/4)[2(A_{r11} + A_{r22})(A_{r12} - A_{r21})S_r(0),$$

$$([\delta][\delta]^{\mathrm{T}})_{22} = (1/4)[2(A_{r12} - A_{r21})^2 S_r(0) + (A_{r11} - A_{r22})^2 S_r(2) + (A_{r12} + A_{r21})^2 S_r(2)],$$

$$([\delta][\delta]^{\mathrm{T}})_{12} = ([\delta][\delta]^{\mathrm{T}})_{21}.$$

Since equation (16a) is independent of $\phi$, the corresponding Fokker–Planck (FK) equation for (16a) is

$$\frac{\partial p}{\partial t} = -\frac{\partial[\varepsilon(m_a)p]}{\partial a} + \frac{\partial^2[\varepsilon([\delta][\delta]^{\mathrm{T}})_{11}p]}{\partial a^2}, \tag{17}$$

where $p$ is henceforth the probability density function and is different from that defined in equation (13).

Setting $T = \varepsilon t$ and substituting $m_a$ and $([\delta][\delta]^{\mathrm{T}})_{11}$ into equation (17), one obtains

$$\frac{\partial p}{\partial T} = r_z \frac{\partial}{\partial a}(a^3 p) + \left(\frac{3\beta}{2} - \alpha\right)\frac{\partial}{\partial a}(ap) + \frac{\beta}{2}\frac{\partial}{\partial a}\left[a^3 \frac{\partial}{\partial a}(p/a)\right], \tag{18}$$

where

$$\alpha = \sigma'(0)\eta + \tfrac{1}{8}[K_r S_r(0) + 3\gamma_r S_r(2)], \qquad \beta = \tfrac{1}{4}[K_r S_r(0) + \gamma_r S_r(2)],$$

$$K_r = 2(A_{r11} + A_{r22})^2, \qquad \gamma_r = (A_{r12} + A_{r21})^2 + (A_{r22} - A_{r11})^2.$$

## 6. BIFURCATIONS

We shall now use some results from reference [15]. These results specifically dealt with the stability of statistical moments of a general single-degree-of-freedom non-linear system. It is summarized in the following lemma.

*Lemma* 1 [15]. Consider the partial differential equation

$$\partial p/\partial T = \tfrac{1}{2}\bar{\beta}_\alpha \partial(a^{1+\Theta}p)/\partial a + \tfrac{1}{2}\beta_0\partial(ap)/\partial a + (\pi/16)S_0\partial[a^3\partial(p/a)/\partial a]/\partial a, \tag{19}$$

where $\bar{\beta}_\alpha > 0$, $S_0 > 0$, $\Theta > -1$, $\beta_0 \in R^1$. If $\Theta > 0$, and we write $\gamma = 1 - 4\beta_0/(\pi S_0)$, then $\langle a^k\rangle \to 0$ as $T\to\infty$ for $\gamma < 0$, and $\langle a^k\rangle \to$constant as $T\to\infty$ for $\gamma > 0$, where $k$ is an integer and $\langle a^k\rangle$ denotes the $k$th order moment of $a$.

Comparing equation (18) with equation (19) one finds that

$$\Theta = 2 > 0, \qquad \overline{\beta}_\alpha = 2r_z, \qquad \beta_0 = 2(\tfrac{3}{2}\beta - \alpha), \qquad S_0 = 8\beta/\pi, \qquad \gamma = (\alpha - \tfrac{1}{2}\beta)/\beta.$$

By the above lemma, when $\gamma < 0$ the $k$th moment $\langle a^k \rangle \to 0$ as $T \to \infty$. For the case in which $\gamma > 0$, $\langle a^k \rangle \to$constant as $T \to \infty$, we obtain $\alpha > \frac{1}{2}\beta$.

By making use of $\alpha$ and $\beta$ in equation (18), we have an inequality relation for $\eta$. From this relation the following two cases can be obtained.

*Case* (*i*). When $\sigma'(0) > 0$, $\eta > \eta_c$, $\langle a^k \rangle \to$constant as $T \to \infty$; $\eta < \eta_c$, $\langle a^k \rangle \to 0$ as $T \to \infty$.

*Case* (*ii*). When $\sigma'(0) < 0$, $\eta > \eta_c$, $\langle a^k \rangle \to 0$ as $T \to \infty$; $\eta < \eta_c$, $\langle a^k \rangle \to$constant as $T \to \infty$; in which the bifurcation point is found at $\alpha = \beta/2$ as

$$\eta_c = -\{1/[4\sigma'(0)]\}\gamma_r S_r(2). \tag{20}$$

When $\sigma'(0) > 0$, one can show that it is associated with a non-linear system having a negative linear damping coefficient, while $\sigma'(0) < 0$ is associated with a non-linear system possessing a positive damping coefficient. Recall that the bifurcation parameter $\eta$ is related to the original bifurcation parameter, $\lambda$, by

$$\lambda = \epsilon\eta,$$

which is the coefficient of the linear damping term in equation (6).

As the $\gamma_r$, which are defined in equation (18) and $S_r(2)$, which is defined in $m_a$ of equation (16a), are positive, the sign of the bifurcation point therefore depends on that of $\sigma'(0)$. By making use of Theorem 1 and equation (20), the bifurcation diagrams for the system with stochastic excitations are as summarized in Figure 1. For direct comparison, bifurcation diagrams of the same system without stochastic excitations are included in Figure 1.

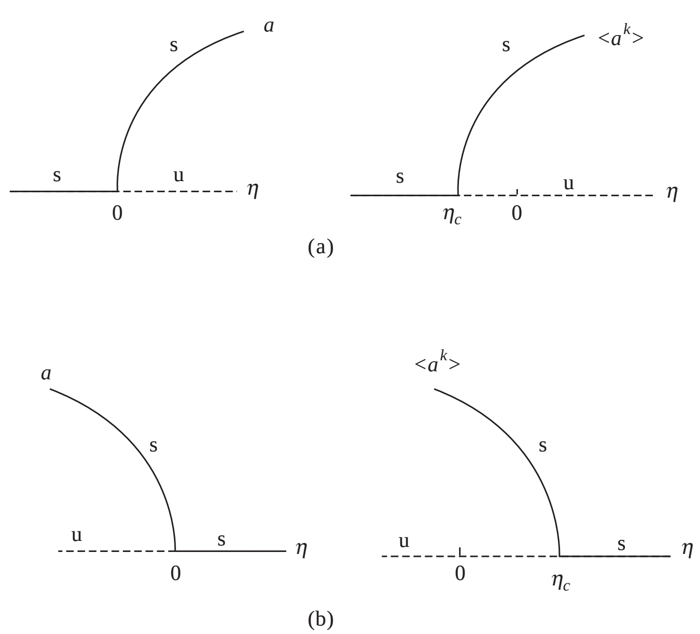


Figure 1. Hopf bifurcations of the single-degree-of-freedom oscillator: "s" and "u" denote stable and unstable solutions, respectively. (a) Supercritical bifurcations, in which $r_z > 0$ and $\sigma'(0) > 0$. (b) Subcritical bifurcations, in which $r_z > 0$ and $\sigma'(0) < 0$. The left-hand parts of (a) and (b) are unperturbed (deterministic), while the right-hand parts are perturbed (stochastic).

Next, we shall obtain the bifurcation points by finding the steady state solution of equation (18). To this end, we set $\partial p/\partial T = 0$. It should be emphasized that, by using the results in either reference [15] or reference [16], one can show that the stationary solution of equation (18) does exist.

Thus, the stationary probability density function can be found as

$$p(a) = 2(r_z/\beta)^{(\alpha/\beta - 1/2)}\Gamma^{-1}(\alpha/\beta - 1/2)a^{2(\alpha/\beta - 1)}\,\mathrm{e}^{-(r_z a^2/\beta)}. \tag{21}$$

The $k$th moment of the amplitude $a$ is obtained as

$$\langle a^k \rangle = \int_0^\infty a^k p(a)\,\mathrm{d}a = (\beta/r_z)^{k/2}\frac{\Gamma[\alpha/\beta + (k-1)/2]}{\Gamma(\alpha/\beta - 1/2)}, \tag{22}$$

provided that $\alpha > \beta/2$.

In equations (21) and (22), when each of the denominators is zero the probability density function and $k$th order stationary moment bifurcate. Thus, the bifurcation points are given by $\alpha = \beta/2$. These are identical to those defined in equation (20), and have been shown to agree with those obtained by using the largest Lyapunov exponent [18]. The notion of bifurcation of non-linear systems under stationary stochastic excitations adopted in the present investigation is similar to that given in references [7, 8] and is in accordance with those defined in references [1–3]. One cannot use $\alpha/\beta = 1$ in equation (21) as a bifurcation point, since at this value the stationary probability density function is not infinite, and is not consistent with the notion of bifurcation in references [1–3].

From the above, we see that the bifurcation points are independent of the order of the moment of the response amplitude. This finding is entirely different from that presented in reference [4], in that there the bifurcation points are dependent of the order of the moment of the response. Our present finding agrees with those for other types of bifurcation, found in references [6, 17]. Furthermore, by making use of $\alpha$ and $\beta$ defined in equation (18) one can show that, even for the case in which $\sigma'(0) < 0$ and $\eta < 0$, the statistical moments in equation (22) are real and approach constant values. In other words, our results for the subcritical case, shown in Figure 1, are correct.

In passing, it should be emphasized that, in the foregoing, the classification of supercritical and subcritical Hopf bifurcations is in accordance with that presented in references [2] and [3].

## 7. CONCLUSIONS

In this note we have presented a method for dealing with bifurcations in two-dimensional non-linear systems excited by stationary stochastic disturbances. Our method is based on the perturbation method, the stochastic averaging of Stratonovich, some results from the theory of singularity and group theory, and results of stability of Brouwers [15]. The latter were applied as a lemma which was employed in the bifurcation analysis. Thus, our method, in which the notion of bifurcation is in accordance with those defined in references [1–3, 7, 8], is different from those proposed in references [4, 11], for example. With our method it was found that the bifurcation points are independent of the order of statistical moments of the amplitude of the response.

With reference to the results presented in Figure 1, it is concluded that, under the stationary stochastic excitation, bifurcation occurs at a negative bifurcation parameter of the system that was originally unperturbed and having a supercritical Hopf bifurcation at a zero bifurcation parameter. Secondly, under the stochastic excitation, bifurcation occurs at a positive bifurcation parameter of a system that was originally unperturbed and has a subcritical Hopf bifurcation at a zero bifurcation parameter.

Finally, it should be mentioned that the bifurcation points for a Van der Pols oscillator obtained by our method agree qualitatively and quantitatively with those derived from using the largest Lyapunov exponent approach [18]. In the latter, the largest Lyapunov exponent of Pardoux and Wihstutz [10] was employed, and it is based on stochastic flow theory, while the presently proposed method hinged around the Markovian property of the response process. Our method, however, is capable of identifying the bifurcation points as well as the type of Hopf bifurcation, whereas the methods based on the largest Lyapunov exponent, and the probability density function (21) or the moment in equation (22), for example, can only locate the bifurcation points.

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## REFERENCES

1. S. N. Chow and J. K. Hale 1982 *Methods of Bifurcation Theory*. New York: Springer-Verlag.
2. J. Guckenheimer and P. Holmes 1983 *Nonlinear Oscillations of Dynamical Systems and Bifurcations of Vector Fields*. New York: Springer-Verlag.
3. M. Golubitsky and D. G. Schaeffer 1985 *Singularity and Groups in Bifurcation Theory, Volume* 1. New York: Springer-Verlag.
4. N. Sri Namachchivaya and S. T. Ariaratnam 1987 *International Journal of Non-linear Mechanics* **22**(5), 363–372. Stochastically perturbed Hopf bifurcation.
5. N. Sri Namachchivaya and H. H. Hilton 1987 *Proceedings of the IUTAM Symposium, Innsbruck, Austria*, 21–26 June, 169–180. Stochastically perturbed bifurcations.
6. C. W. S. To and R. Lin 1989 *Journal of Structural Safety* **6**, 223–231. Bifurcations in a stochastically disturbed nonlinear two-degree-of-freedom system.
7. P. Boxler and L. Arnold 1986 *Report No*. 163, *Institut für Dynamische Systeme, Universitat Bremen*. Stochastische Bifurkationstheorie.
8. P. Boxler 1989 *Journal of Probability Theory and Related Fields* **83**, 509–545. A stochastic version of center manifold theory.
9. S. T. Ariaratnam and W. C. Xie 1988 *Proceedings of Symposium on Stochastic Structural Dynamics, University of Illinois at Urbana–Champaign*, 30 October–November 2. Stochastic perturbation of pitchfork bifurcations.
10. E. Pardoux and V. Wihstutz 1988 *SIAM Journal of Applied Mathematics* **48**, 442–457. Lyapunov exponent and rotation number of two-dimensional linear stochastic systems with small diffusion.
11. M. Doyle, N. Sri Namachchivaya and L. Arnold 1994 *Report No*. 328, *Institut für Dynamische Systeme, Universitat Bremen*. Small noise expansion of moment Lyapunov exponents for two dimensional systems.
12. L. Arnold 1984 *SIAM Journal of Applied Mathematics* **44**, 793–802. A formula connecting sample and moment stability of linear stochastic systems.
13. V. I. Oseledec 1968 *Transactions of the Moscow Mathematical Society* **19**, 197–231. A multiplicative ergodic theorem: Lyapunov characteristic number for dynamical systems.
14. R. L. Stratonovich 1963 *Topics in the Theory of Random Noise, Volumes* 1 *and* 2. New York: Gordon and Breach.
15. J. J. Brouwers 1986 *International Journal of Non-linear Mechanics* **21**, 1–13. Stability of a nonlinearly damped second-order system with randomly fluctuating restoring coefficient.
16. R. Graham and A. Schenzle 1982 *Physical Review A***25**, 1731–1754. Carleman imbedding of multiplicative stochastic processes.

17. W. Wedig 1988 in *Stochastic Structural Dynamics*: *Progress in Theory and Applications* (S. T. Ariaratnam, G. I. Schueller and I. Elishakoff, editors), 315–328. Lyapunov exponents of stochastic systems and related bifurcation problems.
18. C. W. S. To and D. M. Li 1996 *Journal of Shock and Vibration* **3**, 313–320. Largest Lyapunov exponents and bifurcations of stochastic nonlinear systems.